4.5 Solving Systems Using Inverse Matrices

What you should learn

GOAL 1 Solve systems of linear equations using inverse matrices.

GOAL 2 Use systems of linear equations to solve **real-life** problems, such as determining how much money to invest in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as planning a stained glass project in **Ex. 42**.

REAL LIFE

GOAL 1 SOLVING SYSTEMS USING MATRICES

In Lesson 4.3 you learned how to solve a system of linear equations using Cramer's rule. Here you will learn to solve a system using inverse matrices.

> **ACTIVITY**
>
> **Developing Concepts**
>
> **Investigating Matrix Equations**
>
> **1** Write the left side of the matrix equation as a single matrix. Then equate corresponding entries of the matrices. What do you obtain?
>
> $$\begin{bmatrix} 5 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$ **Matrix equation**
>
> **2** Use what you learned in **Step 1** to write the following linear system as a matrix equation.
>
> $2x - y = -4$ **Equation 1**
> $-4x + 9y = 1$ **Equation 2**

In the activity you learned that a linear system can be written as a matrix equation $AX = B$. The matrix A is the coefficient matrix of the system, X is the **matrix of variables**, and B is the **matrix of constants**.

EXAMPLE 1   *Writing a Matrix Equation*

Write the system of linear equations as a matrix equation.

$\quad -3x + 4y = 5$ **Equation 1**

$\quad\quad 2x - y = -10$ **Equation 2**

SOLUTION

$$\overset{A}{\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} 5 \\ -10 \end{bmatrix}}$$

· · · · · · · · · ·

Once you have written a linear system as $AX = B$, you can solve for X by multiplying each side of the matrix by A^{-1} *on the left.*

$\quad\quad AX = B$ **Write original matrix equation.**

$\quad A^{-1}AX = A^{-1}B$ **Multiply each side by A^{-1}.**

$\quad\quad IX = A^{-1}B$ $A^{-1}A = I$

$\quad\quad X = A^{-1}B$ $IX = X$

SOLUTION OF A LINEAR SYSTEM Let $AX = B$ represent a system of linear equations. If the determinant of A is nonzero, then the linear system has exactly one solution, which is $X = A^{-1}B$.

EXAMPLE 2   *Solving a Linear System*

Use matrices to solve the linear system in Example 1.

$$-3x + 4y = 5 \qquad \textbf{Equation 1}$$

$$2x - y = -10 \qquad \textbf{Equation 2}$$

SOLUTION

Begin by writing the linear system in matrix form, as in Example 1. Then find the inverse of matrix A.

$$A^{-1} = \frac{1}{3 - 8} \begin{bmatrix} -1 & -4 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

Finally, multiply the matrix of constants by A^{-1}.

$$X = A^{-1}B = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 \\ -10 \end{bmatrix} = \begin{bmatrix} -7 \\ -4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

▶ The solution of the system is $(-7, -4)$. Check this solution in the original equations.

EXAMPLE 3   *Using a Graphing Calculator*

Use a matrix equation and a graphing calculator to solve the linear system.
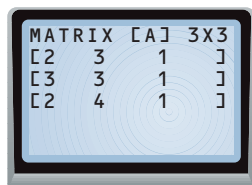
$$2x + 3y + z = -1 \qquad \textbf{Equation 1}$$

$$3x + 3y + z = 1 \qquad \textbf{Equation 2}$$

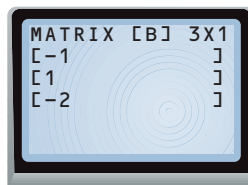$$2x + 4y + z = -2 \qquad \textbf{Equation 3}$$

SOLUTION

The matrix equation that represents the system is $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$.
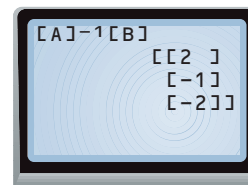
Using a graphing calculator, you can solve the system as shown.

```
MATRIX [A] 3X3
[2    3    1    ]
[3    3    1    ]
[2    4    1    ]
```

```
MATRIX [B] 3X1
[-1            ]
[1             ]
[-2            ]
```

```
[A]-1[B]
          [[2 ]
           [-1]
           [-2]]
```

Enter matrix **A**.   Enter matrix **B**.   Multiply **B** by A^{-1}.

▶ The solution is $(2, -1, -2)$. Check this solution in the original equations.

REAL LIFE **INVESTING**
Each year students across the country in grades 4 through 12 invest a hypothetical $100,000 in stocks to compete in the Stock Market Game. Students can enter their transactions using the internet.

INTERNET **APPLICATION LINK**
www.mcdougallittell.com

GOAL 2 **USING LINEAR SYSTEMS IN REAL LIFE**

EXAMPLE 4 *Writing and Using a Linear System*

🖩 **INVESTING** You have $10,000 to invest. You want to invest the money in a stock mutual fund, a bond mutual fund, and a money market fund. The expected annual returns for these funds are given in the table.

You want your investment to obtain an overall annual return of 8%. A financial planner recommends that you invest the same amount in stocks as in bonds and the money market combined. How much should you invest in each fund?

Investment	Expected return
Stock mutual fund	10%
Bond mutual fund	7%
Money market (MM) fund	5%

SOLUTION

VERBAL MODEL

$$\boxed{\text{Stock amount}} + \boxed{\text{Bond amount}} + \boxed{\text{MM amount}} = \boxed{\text{Total invested}}$$

$$\textbf{0.10} \boxed{\text{Stock amount}} + \textbf{0.07} \boxed{\text{Bond amount}} + \textbf{0.05} \boxed{\text{MM amount}} = \textbf{0.08} \boxed{\text{Total invested}}$$

$$\boxed{\text{Stock amount}} = \boxed{\text{Bond amount}} + \boxed{\text{MM amount}}$$

LABELS

Stock amount = s Money market amount = m

Bond amount = b Total invested = **10,000**

ALGEBRAIC MODEL

$s + b + m = 10,000$ **Equation 1**

$0.10 s + 0.07 b + 0.05 m = 0.08 (10,000)$ **Equation 2**

$s = b + m$ **Equation 3**

First rewrite the equations above in standard form and then in matrix form.

$$\begin{aligned} s + b + m &= 10{,}000 \\ 0.10s + 0.07b + 0.05m &= 800 \\ s - b - m &= 0 \end{aligned} \quad\Rightarrow\quad \begin{bmatrix} 1 & 1 & 1 \\ 0.1 & 0.07 & 0.05 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} s \\ b \\ m \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 800 \\ 0 \end{bmatrix}$$

Enter the coefficient matrix A and the matrix of constants B into a graphing calculator. Then find the solution $X = A^{-1}B$.

```
MATRIX [A] 3X3
[1    1    1   ]
[.1   .07  .05 ]
[1    -1   -1  ]
```

```
MATRIX [B] 3X1
[10000      ]
[800        ]
[0          ]
```

```
[A]−1[B]
      [[5000]
       [2500]
       [2500]]
```
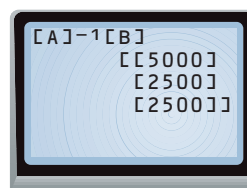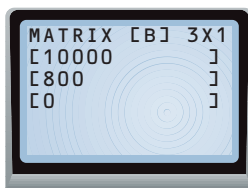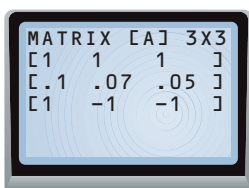
▶ You should invest $5000 in the stock mutual fund, $2500 in the bond mutual fund, and $2500 in the money market fund.

GUIDED PRACTICE

Vocabulary Check ✓

1. What are a matrix of variables and a matrix of constants, and how are they used to solve a system of linear equations?

Concept Check ✓

2. If $|A| \neq 0$, what is the solution of $AX = B$ in terms of A and B?

3. Explain why the solution of $AX = B$ is *not* $X = BA^{-1}$.

Skill Check ✓

Write the linear system as a matrix equation.

4. $x + y = 8$
$2x - y = 6$

5. $x + 3y = 9$
$4x - 2y = 7$

6. $x + y + z = 10$
$5x - y = 1$
$3x + 4y + z = 8$

Use an inverse matrix to solve the linear system.

7. $x + y = 2$
$7x + 8y = 21$

8. $-x - 2y = 3$
$2x + 8y = 1$

9. $4x + 3y = 6$
$6x - 2y = 10$

10. 🌐 **INVESTING** Look back at Example 4 on page 232. Suppose you have $60,000 to invest and you want an overall annual return of 9%. Use the expected annual returns shown to determine how much you should invest in each fund. Assume you are investing as much in stocks as in bonds and the money market combined.

Investment	Expected return
Stock mutual fund	12%
Bond mutual fund	8%
Money market fund	5%

PRACTICE AND APPLICATIONS

WRITING MATRIX EQUATIONS Write the linear system as a matrix equation.

11. $x + y = 5$
$3x - 4y = 8$

12. $x + 2y = 6$
$4x - y = 5$

13. $5x - 3y = 9$
$-4x + 2y = 10$

14. $2x - 5y = -11$
$-3x + 7y = 15$

15. $x + 8y = 4$
$4x - 5y = -11$

16. $2x - 5y = 4$
$x - 3y = 1$

17. $x - 4y + 5z = -4$
$2x + y - 7z = -23$
$-4x + 5y + 2z = 38$

18. $3x - y + 4z = 16$
$2x + 4y - z = 10$
$x - y + 3z = 31$

19. $0.5x + 3.1y - 0.2z = 5.9$
$1.2x - 2.5y + 0.7z = 2.2$
$0.3x + 4.8y - 4.3z = 4.8$

20. $x + z = 9$
$-x - y + 2z = 6$
$2x + 7y - z = -4$

21. $8y - 10z = -23$
$6y - 12z = 14$
$-9x + 5z = 0$

22. $x + y - z = 0$
$2x - z = 1$
$y + z = 2$

SOLVING SYSTEMS Use an inverse matrix to solve the linear system.

23. $3x + y = 8$
$5x + 2y = 11$

24. $x + y = -1$
$11x + 12y = 8$

25. $2x + 7y = -53$
$x + 3y = -22$

26. $7x + 5y = 8$
$4x + 3y = 4$

27. $5x - 7y = 54$
$2x - 4y = 30$

28. $-5x - 7y = -9$
$2x + 3y = 3$

29. $x + 2y = -9$
$-2x - 3y = 14$

30. $2x + 4y = -26$
$2x + 5y = -31$

31. $9x - 5y = 43$
$-2x + 2y = -22$

SOLVING SYSTEMS Use the given inverse of the coefficient matrix to solve the linear system.

32. $2y - z = -2$
$5x + 2y + 3z = 4$
$7x + 3y + 4z = -5$

$$A^{-1} = \begin{bmatrix} -1 & -11 & 8 \\ 1 & 7 & -5 \\ 1 & 14 & -10 \end{bmatrix}$$

33. $x - y - 3z = 9$
$5x + 2y + z = -30$
$-3x - y = 4$

$$A^{-1} = \begin{bmatrix} 1 & 3 & 5 \\ -3 & -9 & -16 \\ 1 & 4 & 7 \end{bmatrix}$$

SOLVING SYSTEMS Use an inverse matrix and a graphing calculator to solve the linear system.

34. $3x + 2y = 13$
$3x + 2y + z = 13$
$2x + y + 3z = 9$

35. $-x + y - 3z = -4$
$3x - 2y + 8z = 14$
$2x - 2y + 5z = 7$

36. $3x + 5y - 5z = 21$
$-4x + 8y - 5z = 1$
$2x - 5y + 6z = -16$

37. $2x + z = 2$
$5x - y + z = 5$
$-x + 2y + 2z = 0$

38. $4x + 3y + z = 14$
$6x + y = 9$
$3x + 5y + 3z = 21$

39. $x + y - 3z = -17$
$2x + z = 12$
$-7x - 2y + z = -11$

40. **SKATING PARTY** You are planning a birthday party for your younger brother at a skating rink. The cost of admission is $3.50 per adult and $2.25 per child, and there is a limit of 20 people. You have $50 to spend. Use an inverse matrix to determine how many adults and how many children you can invite.

41. **DENTAL FILLINGS** Dentists use various amalgams for silver fillings. The matrix shows the percents (expressed as decimals) of powdered alloys used in preparing three different amalgams. Suppose a dentist has 5483 grams of silver, 2009 grams of tin, and 129 grams of copper. How much of each amalgam can be made?

PERCENT ALLOY BY WEIGHT

	Amalgam		
	A	B	C
Silver	0.70	0.72	0.73
Tin	0.26	0.25	0.27
Copper	0.04	0.03	0.00

42. **STAINED GLASS** You are making mosaic tiles from three types of stained glass. You need 6 square feet of glass for the project and you want there to be as much iridescent glass as red and blue glass combined. The cost of a sheet of glass having an area of 0.75 square foot is $6.50 for iridescent, $4.50 for red, and $5.50 for blue. How many sheets of each type should you purchase if you plan to spend $45 on the project?

43. **WALKWAY LIGHTING** A walkway lighting package includes a transformer, a certain length of wire, and a certain number of lights on the wire. The price of each lighting package depends on the length of wire and the number of lights on the wire.

- A package that contains a transformer, 25 feet of wire, and 5 lights costs $20.
- A package that contains a transformer, 50 feet of wire, and 15 lights costs $35.
- A package that contains a transformer, 100 feet of wire, and 20 lights costs $50.

Write and solve a system of equations to find the cost of a transformer, the cost per foot of wire, and the cost of a light. Assume the cost of each item is the same in each lighting package.

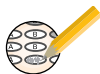**44.** 🌐 **CONSTRUCTION BUSINESS** You are an accountant for a construction business and are planning next year's budget. You have $200,000 to spend on salaries, equipment maintenance, and other general expenses. Based on previous financial records of the business, you expect to spend five times as much on salaries as on equipment maintenance, and you expect general expenses to be 10% of the amount spent on the other two categories combined. Write and solve a system of equations to find the amount you should budget for each category.

Test Preparation

45. **MULTI-STEP PROBLEM** A company sells different sizes of gift baskets with a varying assortment of meat and cheese. A basic basket with 2 cheeses and 3 meats costs $15, a big basket with 3 cheeses and 5 meats costs $24, and a super basket with 7 cheeses and 10 meats costs $50.

   a. Write and solve a system of equations using the information about the basic and big baskets.

   b. Write and solve a system of equations using the information about the big and super baskets.

   c. *Writing* Compare the results from parts (a) and (b) and make a conjecture about why there is a discrepancy.

★ **Challenge**

46. **SOLVING SYSTEMS OF FOUR EQUATIONS** Solve the linear system using the given inverse of the coefficient matrix.

EXTRA CHALLENGE
➤ www.mcdougallittell.com

$$
\begin{aligned}
w + 6x + 3y - 3z &= 2 \\
2w + 7x + y + 2z &= 5 \\
w + 5x + 3y - 3z &= 3 \\
-6x - 2y + 3z &= 6
\end{aligned}
\qquad
A^{-1} = \begin{bmatrix} 40 & -3 & -33 & 9 \\ 1 & 0 & -1 & 0 \\ -39 & 3 & 33 & -8 \\ -24 & 2 & 20 & -5 \end{bmatrix}
$$

MIXED REVIEW

EVALUATING FUNCTIONS Evaluate $f(x)$ or $g(x)$ for the given value of x. (Review 2.7)

$$
f(x) = \begin{cases} \frac{3}{4}x - 8, & \text{if } x \le 8 \\ -x + 6, & \text{if } x > 8 \end{cases}
\qquad
g(x) = \begin{cases} \frac{1}{8}x + 8, & \text{if } x < -1 \\ 2x - 1, & \text{if } x \ge -1 \end{cases}
$$

47. $f(8)$ **48.** $f(11)$ **49.** $f(-2)$ **50.** $f(0)$

51. $g(3)$ **52.** $g(0)$ **53.** $g(-1)$ **54.** $g(-3)$

GRAPHING FUNCTIONS Graph the function and label the vertex. (Review 2.8 for 5.1)

55. $y = |x - 5|$ **56.** $y = |x| + 8$ **57.** $y = -|x - 8| - 9$

58. $y = |x - 5| + 4$ **59.** $y = -|x + 3| + 4$ **60.** $y = -|x + 6| - 2$

FINDING INVERSES Find the inverse of the matrix. (Review 4.4)

61. $\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$ **62.** $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ **63.** $\begin{bmatrix} 8 & 17 \\ -1 & -2 \end{bmatrix}$

64. $\begin{bmatrix} 11 & -5 \\ 3 & -1 \end{bmatrix}$ **65.** $\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$ **66.** $\begin{bmatrix} 6 & -2 \\ 7 & -2 \end{bmatrix}$

QUIZ 2

Find the inverse of the matrix. **(Lesson 4.4)**

1. $\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$　　　**2.** $\begin{bmatrix} -7 & 5 \\ 4 & -3 \end{bmatrix}$　　　**3.** $\begin{bmatrix} -6 & 1 \\ 9 & -3 \end{bmatrix}$　　　**4.** $\begin{bmatrix} 6 & 5 \\ 8 & 7 \end{bmatrix}$

Use an inverse matrix to solve the linear system. **(Lesson 4.5)**

5. $4x + 7y = 24$
$x + 2y = 7$

6. $-9x + 13y = 3$
$2x - 3y = -1$

7. $8x + 7y = 3$
$-2x - 2y = 0$

8. 🌎 **BUYING FLATWARE** The price of flatware varies depending on the number of place settings you buy as well as other items included in the set. Suppose a set with 4 place settings costs $142 and a set with 8 place settings and a serving set costs $351. Find the cost of a place setting and a serving set. Assume that the cost of each item is the same for each flatware set. **(Lesson 4.5)**

MATH & History

Systems of Equations

**APPLICATION LINK**
INTERNET
www.mcdougallittell.com

THEN

OVER 2000 YEARS AGO, a method for solving systems of equations using rectangular arrays of counting rods was presented in *Nine Chapters on the Mathematical Art*, an early Chinese mathematics text. A problem from this book is given below.

Three bundles of top-grade ears of rice, two bundles of medium-grade ears of rice, and one bundle of low-grade ears of rice make 39 dou (of rice by volume); two bundles of top-grade ears of rice, three bundles of medium-grade ears of rice, and one bundle of low-grade ears of rice make 34 dou; one bundle of top-grade ears of rice, two bundles of medium-grade ears of rice, and three bundles of low-grade ears of rice make 26 dou. How many dou are there in a bundle of each grade of rice?

Top-grade ears of rice
Medium-grade ears of rice
Low-grade ears of rice

Total produce
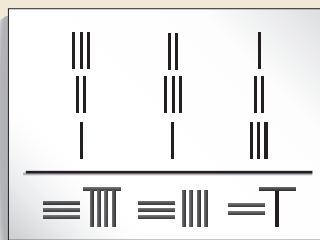
1. Use matrices to organize the given information and solve the problem.

2. How is the arrangement of counting rods similar to your matrices? How is it different?

NOW

TODAY, computers use matrices representing systems with many variables to solve complicated problems like predicting weather and designing aircraft.

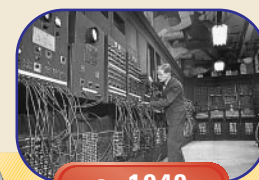Great Wall of China planned

c. 220 B.C.

A.D. 50
Chinese method of solving systems described

Mechanical calculator, France
1642

c. 1940
Electronic computers, USA